**Lemma 1.** *An optimal $n$-town is convex.*

The following proof holds in any dimension and with any norm for measuring the distance between points.

*Proof.* Let $S$ be an $n$-town which is not convex. Then there are points $x_1, x_2, \ldots, x_k \in S$ and a grid point $x \notin S$ such that $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k$ for some $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ with $\sum \lambda_i = 1$. Because every norm is a convex function, and the sum of convex functions is again convex, the function $f_S(x) = c(x, S) = \sum_{s \in S} \|x - s\|_1$ is convex. Therefore,

$$f_S(x) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_k f(x_k),$$

which implies $f_S(x) \leq f_S(x_i)$ for some $i$. Using Lemma X we get

$$c((S \setminus x_i) \cup x) = c(S) - f_S(x_i) + f_S(x) - \|x - x_i\|_1 < c(S). \qquad \square$$

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Reviewer: It may be better to start with "Let $S$ be an *optimal* $n$-town which is not convex." and end with something like "This contradicts the assumption that $S$ is optimal."

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*Proof.* Let $S$ be an optimal $n$-town which is not convex. Then there are points $x_1, x_2, \ldots, x_k \in S$ and a grid point $x \notin S$ such that $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k$ for some $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ with $\sum \lambda_i = 1$. Because every norm is a convex function, and the sum of convex functions is again convex, the function $f_S(x) = c(x, S) = \sum_{s \in S} \|x - s\|_1$ is convex. Therefore,

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$$c((S \setminus x_i) \cup x) = c(S) - f_S(x_i) + f_S(x) - \|x - x_i\|_1 < c(S).$$

This contradicts the assumption that $S$ is optimal. $\square$

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*Proof.* We prove that a non-convex $n$-town $S$ cannot be optimal. Take a grid point $x \notin S$ in the convex hull of $S$. Then there are points $x_1, x_2, \ldots, x_k \in S$ such that $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k$ for some $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ with $\sum \lambda_i = 1$. Because every norm is a convex function, and the sum of convex functions is again convex, the function $f_S(x) = c(x, S) = \sum_{s \in S} \|x - s\|_1$ is convex. Therefore,

$$f_S(x) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_k f(x_k),$$

which implies $f_S(x) \leq f_S(x_i)$ for some $i$. Using Lemma X we get

$$c((S \setminus x_i) \cup x) = c(S) - f_S(x_i) + f_S(x) - \|x - x_i\|_1 < c(S).$$

This means that $S$ is not optimal. $\square$