LEMMA 1. *Consider two planes in* $\mathbb{R}^3$ *with unit normals* u^+ *and* u^-. *Assume that* u^+ *and* u^- *enclose an angle at most* $\varepsilon \in (0, \pi/2)$ *with the* xy*-plane, and the angle between them lies in* $[\beta, \pi - \beta]$, *where* $0 < \beta \leqq \pi/2$. *Then their intersection line encloses an angle at most*

$$\delta := \arcsin \frac{\sin \varepsilon}{\sin(\beta/2)}$$

with the z*-axis, provided that* $\varepsilon \leqq \beta/2$. *This inequality is sharp.*

PROOF. We choose a new coordinate system in the following way. The intersection line becomes the vertical axis, and the two normal vectors $u^+, u^- \in \mathbb{S}^2$ lie in the horizontal plane, enclosing an angle $\beta' \in [\beta, \pi - \beta]$ with each other. In the new coordinate system, the original North Pole becomes $n = (n_1, n_2, n_3) \in \mathbb{S}^2$.

By hypothesis,

$$\langle n, u^- \rangle, \langle n, u^+ \rangle \in [-\sin \varepsilon, \sin \varepsilon]. \tag{5}$$

We want to conclude that

$$\left| \left\langle (0,0,1), n \right\rangle \right| = |n_3| \geqq \cos \delta, \tag{6}$$

i.e., that

$$\sqrt{n_1^2 + n_2^2} \leqq \sin \delta. \tag{7}$$

The points $(n_1, n_2) \in \mathbb{R}^2$ (projections of n to the xy-plane) for n satisfying (5) form a rhomb of height $2 \sin \varepsilon$ and angles $\beta', \pi - \beta'$. A farthest point of this rhomb from $(0,0)$ is one of the vertices and its distance from $(0,0)$ is $\max \left\{ (\sin \varepsilon)/\sin(\beta'/2), (\sin \varepsilon)/\cos(\beta'/2) \right\} \leqq (\sin \varepsilon)/\sin(\beta/2) = \sin \delta$. That is, (7), or equivalently, (6) holds and both are sharp inequalities. Hence, the inequality of the lemma holds and it is sharp. $\square$

LEMMA 2. *Consider a convex polyhedron* $P \subset \mathbb{R}^3$ *with facet areas* $S_1, \ldots, S_m$. *Assume that its facet outer normals enclose an angle at most* ε *with the* xy*-plane and the angle between any two of them lies in* $[\beta, \pi - \beta]$, *where* $0 < \beta \leqq \pi/2$. *Then its volume is bounded by*

$$V(P) \leqq 2^{-1/4} \pi^{-1} \cdot \left(\sum_{i=1}^m S_i^{3/4} \right)^2 \cdot \left(\frac{\sin \varepsilon}{\sin(\beta/2)} \right)^{1/2},$$

if $(\sin \varepsilon)/\sin(\beta/2) \leqq 1/\sqrt{2}$.

PROOF. We denote by $s_i(z)$ the length of the horizontal cross-section of the i-th facet at height z, and by $s_i^{\max}$ the maximum length of such a horizontal cross-section. Let h_i be the "height" of the i-th face: the difference between the maximum and the minimum z-coordinates of its points. Let h_i' be the "tilted height" of this facet in its own plane, i.e., the height when the plane is rotated into vertical position about one of its horizontal cross-sections.

Since $(\sin\varepsilon)/\sin(\beta/2) < 1$, we have by Lemma 1 that P has no horizontal edges. Therefore, using the quantity δ introduced in Lemma 1, we get

$$s_i^{\max} \leqq h_i \cdot \tan\delta. \tag{8}$$

Namely, from the minimal z-coordinate – where $s_i(z) = 0$ – $s_i(z)$ can increase only with a speed at most $2\tan\delta$ $(< \infty)$ to reach its maximal value $s_i^{\max}$. This is clear for a vertical face, and for a nonvertical face the speed is even smaller. Observe that the i-th facet lies in an upwards circular cone with vertex the lowest point of the i-th facet and directrices enclosing an angle δ with the z-axis. From the maximal value it must decrease again with speed at most $2\tan\delta$ till 0 at the maximal z-coordinate.

Therefore, using for (10) inequality (8),

$$S_i \geqq s_i^{\max} h_i'/2 \geqq s_i^{\max} h_i/2 \tag{9}$$

$$\geqq (s_i^{\max})^2/(2\tan\delta). \tag{10}$$

This gives

$$s_i^{\max} \leqq \sqrt{2S_i \tan\delta}. \tag{11}$$

These relations allow us to bound the volume $V(P)$ as follows, by using the isoperimetric inequality on each horizontal slice.

$$V(P) = \int_{-\infty}^{\infty} (\text{area of cross-section of } P \text{ at height } z)\, dz$$

$$\leqq \int_{-\infty}^{\infty} \left(\sum_{i=1}^{m} s_i(z) \right)^2 dz/(4\pi)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{-\infty}^{\infty} s_i(z) s_j(z)\, dz/(4\pi)$$

$$\leqq \sum_{i=1}^{m} \sum_{j=1}^{m} s_i^{\max} s_j^{\max} \min\{h_i, h_j\}/(4\pi) \tag{12}$$

$$\leqq \sum_{i=1}^{m} \sum_{j=1}^{m} s_i^{\max} s_j^{\max} \sqrt{h_i h_j}/(4\pi)$$

$$= \left(\sum_{i=1}^{m} s_i^{\max} \sqrt{h_i} \right)^2 /(4\pi)$$

$$= \left(\sum_{i=1}^{m} \sqrt{s_i^{\max}} \sqrt{s_i^{\max} h_i} \right)^2 \Big/ (4\pi)$$

$$\leqq \left(\sum_{i=1}^{m} (2S_i \tan \delta)^{1/4} \sqrt{2S_i} \right)^2 \Big/ (4\pi) \tag{13}$$

$$= \frac{\left(\sum_{i=1}^{m} S_i^{3/4} \right)^2}{\sqrt{2}\pi} \cdot \sqrt{\frac{(\sin \varepsilon)/\sin(\beta/2)}{\sqrt{1 - (\sin^2 \varepsilon)/\sin^2(\beta/2)}}} \tag{14}$$

$$\leqq \frac{\left(\sum_{i=1}^{m} S_i^{3/4} \right)^2}{\sqrt{2}\pi} \cdot \sqrt{\frac{\sqrt{2} \sin \varepsilon}{\sin(\beta/2)}} .$$

The first inequality uses the isoperimetric inequality. The second inequality (12) bounds the integral by an upper bound of the non-negative integrand times the length of the interval where the integrand is positive. For (13), we have used (9) and (11). To obtain (14), we have used Lemma 1. The last inequality simplifies the denominator under the assumption $(\sin \varepsilon)/\sin(\beta/2) \leqq 1/\sqrt{2}$ of the lemma. □

4.6. Third proof of Theorem 2 for $n = 3$ dimensions

As in the first proof, we use Minkowski's Theorem F′. We want to apply Lemma 2, making ε small. Thus, we must let the normal vectors with given lengths S_i converge to the xy-plane, keeping their sum to be 0. Moreover, the linear span of the outer unit facet normals should be $\mathbb{R}^3$. Then we apply Minkowski's Theorem F′. In the limiting configuration the normals will lie

Now, since $\sin^2 \varepsilon \leqq b^2/\left[2(n-1)^3\right]$, the angle δ from Lemma 3 lies in $(0, \pi/2)$. Hence, by Lemma 3, P has no horizontal edges, and thus, also no horizontal k-faces for any $k \in \{1, \ldots, n-2\}$. Therefore, once more by Lemma 3, we know that every facet is contained in two rotationally symmetric cones with $(n-1)$-balls as bases. One cone has its apex at the unique lowest point of this facet and extends upwards from there. Its axis is vertical (parallel to the x_n-direction), and the directrices enclose an angle δ with the x_n-axis. The other cone extends downwards from the highest point of the facet and has a vertical axis and directrices enclosing an angle δ with the x_n-axis. We use the upwards cone from the minimal height till the arithmetic mean of the minimal and maximal heights. We use the downward cone for the other half of the vertical extent of the facet. By this argument, we can bound the maximum cross-section area $s_i^{\max}$ of the i-th facet as follows.

$$s_i^{\max} \leqq \big((h_i/2) \cdot \tan \delta\big)^{n-2} \cdot \kappa_{n-2}. \tag{17}$$

(From the minimal height till the arithmetic mean of the minimal and maximal heights we have the following. Any horizontal cross-section of the cone is contained in some $(n-1)$-ball of radius at most $R := (h_i/2) \cdot \tan \delta$. Thus, any horizontal cross-section of the facet lies inside the intersection of its own affine hull with the upwards cone. That is, it lies in the intersection of an $(n-2)$-dimensional affine subspace with a cone whose base is an $(n-1)$-ball of radius at most R. Hence, this horizontal cross-section lies inside some $(n-2)$-ball of radius at most R. A similar argument holds for the downward cone.) Moreover, we also have

$$S_i \geqq s_i^{\max} h_i'/(n-1) \geqq s_i^{\max} h_i/(n-1). \tag{18}$$

Let us rewrite (17) and (18) as follows.

$$h_i^{-(n-2)} \cdot s_i^{\max} \leqq \big((\tan \delta)/2\big)^{n-2} \cdot \kappa_{n-2} \tag{19}$$

$$h_i \cdot s_i^{\max} \leqq (n-1) S_i. \tag{20}$$

We multiply the $1/\left[(2n-2)(n-2)\right]$-th power of (19) with the $n/(2n-2)$-th power of (20) to get an inequality that we will need.

$$\begin{aligned} &(s_i^{\max})^{(n-1)/(2n-4)} \sqrt{h_i} \\ &\quad \leqq \big((\tan \delta)/2\big)^{1/(2n-2)} \cdot (\kappa_{n-2})^{1/\left[(2n-2)(n-2)\right]} \cdot \big((n-1)S_i\big)^{n/(2n-2)}. \end{aligned} \tag{21}$$

Let $K := \big[(n-1)^{n-1}\kappa_{n-1}\big]^{-1/(n-2)}$ denote the constant of the isoperimetric inequality in $n-1$ dimensions:

$$V_{n-1}(C) \leqq K \cdot \big(V_{n-2}(\partial C)\big)^{(n-1)/(n-2)} \tag{22}$$

(for $C \subset \mathbb{R}^{n-1}$). Now we can bound the volume as follows.

$$\begin{aligned}
V(P) &= \int_{-\infty}^{\infty} \big[(n-1)\text{-volume of the cross-section of } P \text{ at height } x_n\big]\, dx_n \\
&\leqq \int_{-\infty}^{\infty} \left[\left(\sum_{i=1}^{m} s_i(x_n)\right)^{(n-1)/(2n-4)}\right]^2 dx_n \cdot K \\
&\leqq \int_{-\infty}^{\infty} \left[\sum_{i=1}^{m} s_i(x_n)^{(n-1)/(2n-4)}\right]^2 dx_n \cdot K \\
&= \int_{-\infty}^{\infty} \sum_{i=1}^{m}\sum_{j=1}^{m} s_i(x_n)^{(n-1)/(2n-4)} s_j(x_n)^{(n-1)/(2n-4)}\, dx_n \cdot K \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{\infty} s_i(x_n)^{(n-1)/(2n-4)} s_j(x_n)^{(n-1)/(2n-4)}\, dx_n \cdot K \\
&\leqq \sum_{i=1}^{m}\sum_{j=1}^{m} (s_i^{\max})^{(n-1)/(2n-4)} (s_j^{\max})^{(n-1)/(2n-4)} \min\{h_i, h_j\} \cdot K \qquad (23) \\
&\leqq \sum_{i=1}^{m}\sum_{j=1}^{m} (s_i^{\max})^{(n-1)/(2n-4)} (s_j^{\max})^{(n-1)/(2n-4)} \sqrt{h_i h_j} \cdot K \\
&\leqq (\tan\delta)^{1/(n-1)} \cdot 2^{-1/(n-1)} \cdot (\kappa_{n-2})^{1/[(n-1)(n-2)]} \cdot (n-1)^{n/(n-1)} \qquad (24) \\
&\qquad \cdot \left(\sum_{i=1}^{m} S_i^{n/(2n-2)}\right)^2 \cdot K \\
&= \text{const}_n \cdot \left(\sum_{i=1}^{m} S_i^{n/(2n-2)}\right)^2 \left(\frac{(n-1)^{3/2}(\sin\varepsilon)/b}{\sqrt{1-(n-1)^3(\sin^2\varepsilon)/b^2}}\right)^{1/(n-1)} \qquad (25)
\end{aligned}$$

$$\leqq \text{const}'_n \cdot \left(\sum_{i=1}^{m} S_i^{n/(2n-2)} \right)^2 \cdot \left(\frac{\sin \varepsilon}{b} \right)^{1/(n-1)} .$$

The first inequality uses the isoperimetric inequality (22). The second inequality uses the concavity of the function $t^{(n-1)/(2n-4)}$ for $t \in [0, \infty)$ and its vanishing at $t = 0$. (Observe that $0 < (n-1)/(2n-4) \leqq 1$.) Inequality (23), as in (12), bounds the integral of a non-negative function by an upper bound of the integrand times the length of the interval where the integrand is positive. For (24), we have used the bound (21) that we derived above. Inequality (25) uses the bound δ from Lemma 3. Finally, by hypothesis, the expression under the square root in the denominator of (25) is bounded below by $1 - (n-1)^3 (\sin^2 \varepsilon)/b^2 \geqq 1/2$. We have therefore established the claimed upper bound.

Now we give the example for the lower bound for $n \geqq 3$ and $m \geqq 2n$. Let $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 \in (0, \pi/4)$ will be chosen later. Let us write $\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}$. Let $T^+, T^- \subset \mathbb{R}^{n-1}$ be regular $(n-1)$-simplices circumscribed about the unit ball B^{n-1} of $\mathbb{R}^{n-1}$. Put them in such a general position w.r.t. each other so that any $n-1$ of their altogether $2n$ facet outer normals linearly span $\mathbb{R}^{n-1}$. Let $n \leqq m^+, m^-$ and $m = m^+ + m^-$. Let $R^\pm$ be obtained from $T^\pm$ by intersecting it still with $m^\pm - n$ closed halfspaces in $\mathbb{R}^{n-1}$, all containing B^{n-1}, with their boundaries touching B^{n-1}. Then

$$B^{n-1} \subset R^\pm \subset T^\pm \subset (n-1) B^{n-1}.$$

Let the altogether $m = m^+ + m^-$ facet outer unit normals of R^+ and R^- satisfy the same condition of general position as above. Namely, any $n-1$ of them linearly span $\mathbb{R}^{n-1}$. Let $b > 0$ be the minimum of the $(n-1)$-volumes of the $(n-1)$-parallelotopes spanned by any $n-1$ of these altogether m facet outer unit normals.

Observe that for $n = 3$ and $m \geqq 2$, the largest value of b is $\sin(\pi/m)$ – if we do not begin the construction with two regular triangles but allow any m facet outer unit normals in $\mathbb{S}^{n-2} = \mathbb{S}^1$. For $n > 3$, the maximal value of b can be bounded from above as follows – again not beginning with two regular simplices, but allowing any m facet outer unit normals in $\mathbb{S}^{n-2}$. Let us choose altogether $n-1$ outer unit normal vectors of R^+ and R^-, say, $u_1, \ldots, u_{n-1} \in \mathbb{S}^{n-2}$. We have

$$\frac{\left| \det(u_1, \ldots, u_{n-1}) \right|}{(n-1)!}$$
$$= V_{n-2}\big(\text{conv}\{u_1, \ldots, u_{n-1}\}\big) \cdot \text{dist}\left(0, \text{aff}\{u_1, \ldots, u_{n-1}\}\right) / (n-1)$$
$$\leqq V_{n-2}\big(\text{conv}\{u_1, \ldots, u_{n-1}\}\big) / (n-1).$$