MATHEMATICAL WRITING, Günter Rote, WS 2014/15.
Exercise 1; Oct 23, 2014.

Lemma 1. *Let* $1 \leq i \leq l$, $m_i \geq 2$ *and* $x_{i,1} \leq x_{i,2} \leq \ldots \leq x_{i,m_i-1}$ *and* $x_{l+1,1}$ *positive integers. For* $1 \leq i \leq l$ *we denote* $y_i = \sum_{j=1}^{m_{i-1}} x_{i,j}$ *and* $y_{l+1} = x_{l+1,1}$*. If for every* $1 \leq i \leq l$ *and every* $1 \leq j \leq m_{i-1}$, $x_{i,j} \leq \sum_{k=i+1}^{l+1} y_k$ *and* $y_{l+1} = 1$ *and* z *is a nonnegative integer with* $z \leq N = \sum x_{i,j} = \sum_{k=1}^{l+1} y_k$*, then there is a subset* Z *of the* $x_{i,j}$*'s which sums to* z*.*

Proof. Let us consider the following greedy algorithm. We process the series in the order $x_{1,1}, x_{1,2}, \ldots, x_{1,m_1-1}, x_{2,1}, x_{2,2}, \ldots, x_{2,m_2-1}, \ldots, x_{l,1}, x_{l,2}, \ldots, x_{l,m_i-1}$, and we add an element $x_{i,j}$ to Z if and only if the sum together with it is at most z. We show that this algorithm gives the desired subset at the end.

For that we show that after processing $x_{1,1}, x_{1,2}, \ldots, x_{1,m_1-1}, x_{2,1}, x_{2,2}, \ldots, x_{2,m_2-1}, \ldots, x_{i,1}, x_{i,2}, \ldots, x_{i,m_i-1}$ the sum is in the interval $[z - (y_{i+1} + \ldots + y_{l+1}), z]$ (this is enough, as at the end $i = l + 1$ and this interval has length 0).

During the process the sum is always at most z by definition of the process, thus we only need to prove that the sum is at least $z - (y_{i+1} + \ldots + y_{l+1}$ at the above given time. Note that during the process the sum is monotonely increasing as we only add $x_{i,j}$'s to the sum.

We use induction on i. In case $i = 1$, either we can add every $x_{1,j}$ to Z, or not. If we can, the sum is $y_1 = N - (y_2 + \ldots + y_{l+1}) \geq z - (y_2 + \ldots + y_{l+1})$. If we cannot, it means that a certain $x_{1,j}$ would push the sum over z. But then without that $x_{1,j}$ the sum when processing $x_{1,j}$ was already greater than $z - x_{1,j} \geq z - (y_2 + \ldots + y_{l+1})$.

The induction step goes similarly. Suppose that the sum after $x_{1,1}, x_{1,2}, \ldots, x_{1,m_1-1}, x_{2,1}, x_{2,2}, \ldots, x_{2,m_2-1}, \ldots, x_{i-1,1}, x_{i-1,2}, \ldots, x_{i-1,m_{i-1}-1}$ is in the interval $[z - (y_i + \ldots + y_{l+1}), z]$, and consider $x_{i,1}, x_{i,2}, \ldots, x_{i,m_i-1}$. If we can add all of them to Z, then the sum increases by y_i, hence it is at least $z - (y_i + \ldots + y_{l+1}) + y_i = z - (y_{i+1} + \ldots + y_{l+1})$. If not, then again a certain $x_{i,j}$ would push it over z, which means the sum was already greater than $z - x_{i,j} \geq z - (y_{i+1} + \ldots + y_{l+1})$.

Thus by induction before processing $x_{l+1,1} = 1$ we have a sum in the range $[z - y_{l+1}, z] = [z - 1, z]$. If the sum is z then we are done, otherwise it is $z - 1$, and we add $x_{l+1,1} = 1$ to the sum to make it z. □