32. Static rigidity and Dehn's theorem

This is the first of two sections where we prove *Dehn's theorem*, an infinitesimal version of the *Cauchy rigidity theorem*. We include two proofs: a variation on Cauchy's original proof in Section 26 and Dehn's original proof.

32.1. **Who needs rigidity?** In the next two sections we introduce two new concepts, the *static* and the *infinitesimal rigidity* of convex polyhedra, which turn out to be equivalent to each other and imply continuous rigidity. These ideas are crucial in modern rigidity theory; their natural extensions to general frameworks (of bars, cables and struts) were born out of these considerations and have a number of related properties. While we spend no time at all on these extensions, we find these ideas useful in discussions on Cauchy's and Gluck's theorems.

To summarize the results in the next two sections, we show that Gluck's Theorem 31.3 follows from *Dehn's lemma* on the determinant of a *rigidity matrix*, which we also introduce. We also show that the continuous rigidity also follows from Dehn's lemma. We then present three new proofs of Dehn's lemma, all without the use of the Cauchy theorem, as well as one extra proof of continuous rigidity of convex polyhedra. As the reader shall see, all this is done and motivated by the two rigidity concepts.

32.2. **Loading the edges.** To define the static rigidity, we need to extract the key ingredient in the proof of Gluck's theorem we presented in the previous section.

Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of a plane triangulation $\Gamma = (V, E)$, and denote by $f : V \to \mathbb{R}^3$ its planted realization. Let E be a set of ordered pairs: if $(v_i, v_j) \in E$, then $(v_j, v_i) \in E$ as well. Now, for every edge $e = (v_i, v_j) \in E$, denote by

$$\boldsymbol{e}_{ij} \, = \, \overrightarrow{f(v_i) f(v_j)} \, = \, (x_j - x_i, y_j - y_i, z_j - z_i) \in \mathbb{R}^3$$

the corresponding edge vector in the realization. In this notation, $\boldsymbol{e}_{ij} = -\boldsymbol{e}_{ji}$, for all $(v_i, v_j) \in E$. The set of scalars $\{\lambda_{ij} \in \mathbb{R},\, (v_i, v_j) \in E\}$ is said to be an *edge load* if $\lambda_{ij} = -\lambda_{ji}$, $\lambda_{12} = \lambda_{13} = \lambda_{23} = 0$, and

$$\sum_{j \,:\, (v_i, v_j) \in E} \lambda_{ij} \boldsymbol{e}_{ij} = 0, \quad \text{for all } \ i \in [n].$$

We say that a planted realization $f : V \to \mathbb{R}^3$ of (V, E) defining the polytope is *statically rigid* if there is no nonzero static load $\{\lambda_{ij}\}$. Finally, a simplicial convex polytope P with graph $\Gamma = (V, E)$ is *statically rigid* if so is the planted realization of Γ obtained by a rigid motion of P. The following result is the key result of this section.

Theorem 32.1 (Dehn's theorem; static rigidity of convex polytopes)**.** *Every simplicial convex polytope in* $\mathbb{R}^3$ *is statically rigid.*

We already proved this result in a different language. To see this, consider a matrix $\mathcal{R}_\Gamma$ with rows $\mathcal{R}_\Gamma^{(ij)}$ corresponding to edges $(v_i, v_j) \in E$, written in lexicographical order:

$$\mathcal{R}_\Gamma^{(ij)} \, = \, (\ldots, x_i - x_j, y_i - y_j, z_i - z_j, \ldots, x_j - x_i, y_j - y_i, z_j - z_i, \ldots).$$

The matrix $\mathcal{R}_\Gamma$ is called the *rigidity matrix*. Now observe, the Jacobian $J(\cdot)$ is a determinant of the matrix with the following rows:

$$dF_{ij} = \left(\ldots, \frac{\partial F_{ij}}{\partial x_r}, \frac{\partial F_{ij}}{\partial y_r}, \frac{\partial F_{ij}}{\partial z_r}, \ldots\right) = 2\,\mathcal{R}_\Gamma^{(ij)}.$$

We showed that the Jacobian $J(\cdot) = 2^m \det \mathcal{R}_\Gamma \neq 0$, when evaluated at a planted realization $f : \Gamma \to \mathbb{R}^3$ defined by a convex polytope P. Thus, there is no nonzero linear combination of the rows of the rigidity matrix, with coefficients λ_{ij} as above. Interpreting the set of coefficients $\{\lambda_{ij}\}$ as the edge load, we obtain the statement of Dehn's theorem.

In the opposite direction, given Theorem 32.1, we obtain that $\det \mathcal{R}_\Gamma \neq 0$. Therefore, the Jacobian is nonzero, which in turn implies Gluck's theorem without the use of the Cauchy theorem. To conclude this discussion, the static rigidity of convex polytopes is equivalent to the following technical statement.

Lemma 32.2 (Dehn). *Let $P \subset \mathbb{R}^3$ be a simplicial convex polytope with a graph $\Gamma = (V, E)$. Then the rigidity matrix $\mathcal{R}_\Gamma$ is nonsingular.*

In the following two subsections we present three independent proofs of Dehn's lemma, all (hopefully) easier and more elegant than any of the previous proofs of the Cauchy theorem. Until then, let us make few more comments.

First, let us show that Dehn's lemma implies the (continuous) rigidity of convex polytopes (Corollary 26.7). Indeed, consider the m-dimensional space W of planted realizations of $\Gamma = (V, E)$. The space W is mapped onto the m-dimensional space of all length functions, and the determinant $J(\cdot) = 2^m \det \mathcal{R}_\Gamma$ is nonzero at convex realizations. Therefore, in a small neighborhood of a convex realization the edge lengths are different, and thus a simplicial convex polytope is always rigid.

Our second observation is that the Cauchy theorem is more powerful than Theorem 32.1. To see this, recall the Cauchy–Alexandrov theorem on uniqueness of convex polyhedral surfaces (Theorem 27.6). This immediately implies the rigidity of these surfaces[77]. On the other hand, such realizations are not necessarily statically rigid, as the example in Figure 32.1 shows. Here we make arrows in the directions with positive coefficients which are written next to the corresponding edges.

32.3. **Proof of Dehn's lemma via sign changes.** This proof goes along the very same lines as the traditional proof of the Cauchy theorem (see Section 26.3). We first show that the edge load $\{\lambda_{ij}\}$ gives a certain assignment of signs on edges, then prove the analogue of the sign changes lemma (Lemma 26.4), and conclude by using the sign counting lemma (Lemma 26.5).

Proof of Dehn's lemma. Consider an edge load $\{\lambda_{ij} \mid (v_i, v_j) \in E\}$ on the edges in P. To remove ubiquity, consider only coefficients λ_{ij} with $i < j$. Let us label the edge $(v_i, v_j) \in E$, $i < j$, with $(+)$ if $\lambda_{ij} > 0$, with $(-)$ if $\lambda_{ij} < 0$, and with (0) if $\lambda_{ij} = 0$.

[77] One has to be careful here: this only proves rigidity in the space of *convex realizations*. In fact, the continuous rigidity holds for *all* non-strictly convex realizations; this is a stronger result due to Connelly (see [Con5])

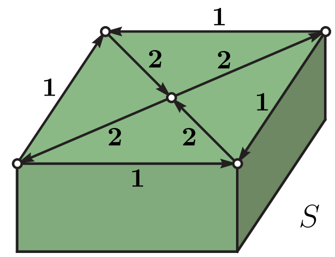

FIGURE 32.1. Stresses on a non-strictly convex polyhedral surface S show that it is not statically rigid.

Lemma 32.3 (Static analogue of the sign changes lemma)**.** *Unless all labels around a vertex are zero, there are at least four sign changes.*

By Lemma 26.5, we conclude that all labels must be zero. This proves Dehn's lemma modulo Lemma 32.3. $\square$

Proof of Lemma 32.3. Denote by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$ the edge vectors of edges leaving vertex w of a convex polytope P. We assume that w is at the origin and the edge vectors are written in cyclic order. Suppose we have a nonzero linear combination

$$\boldsymbol{u} := \lambda_1 \boldsymbol{e}_1 + \ldots + \lambda_k \boldsymbol{e}_k = 0.$$

Denote by H any generic hyperplane supporting P at w, i.e., a hyperplane containing w, such that all vectors $\boldsymbol{e}_i$ lie in the same half-space. If there are no sign changes, i.e., $\lambda_1, \ldots, \lambda_k \geq 0$ or $\lambda_1, \ldots, \lambda_k \leq 0$. Then their linear combination $\boldsymbol{u}$ is also in the same half-space unless all $\lambda_i = 0$, a contradiction.

Suppose now that there are two sign changes, for simplicity $\lambda_1, \ldots, \lambda_i \geq 0$ and $\lambda_{i+1}, \ldots, \lambda_k \leq 0$. Denote by H a hyperplane which contains vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_i$ in a half-space H_+, and $\boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_k$ in the other half-space H_-. Then the linear combi-

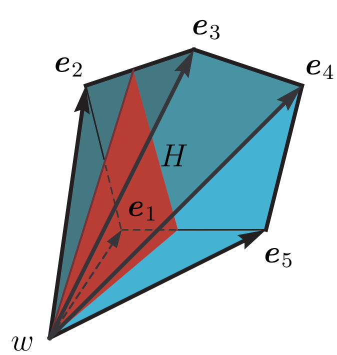

FIGURE 32.2. Hyperplane H separating edges in a polytope P.

nation $\boldsymbol{u}$ is in H_+ unless all $\lambda_i = 0$, another contradiction (see Figure 32.2). Thus, there are at least four sign changes unless all labels are zero. $\square$

32.4. **Proof of continuous rigidity from the angular velocity equation.** Before we continue with other proofs of Dehn's lemma, let us show how continuous rigidity (Corollary 26.7) follows from the angular velocity equation (Lemma 28.2), which was proved in Section 29 by a simple independent argument. The proof will be almost completely the same as the above proof of Dehn's lemma.

Proof of Corollary 26.7 modulo Lemma 28.2. Consider the angular velocity equation for each vertex and assign labels to all edges according to the signs of derivatives $\theta_e'(t)$. By the proof of Lemma 32.5, either all labels around a vertex are zero, or there are at least four sign changes. Now use the sign counting lemma (Lemma 26.5) to conclude that all labels bust be zero, i.e., all derivatives are zero (more precisely, all left and right derivatives at each point are zero, which is equivalent). Thus the dihedral angles remain constant under the continuous deformation, and the deformation itself is a rigid motion. $\square$

32.5. **Graph-theoretic proof of Dehn's lemma.** Let $\Gamma = (V, E)$ be a plane triangulation, and let $\mathcal{R}_\Gamma(\ldots, x_r, y_r, z_r, \ldots)$ be the rigidity matrix defined above. To prove Dehn's lemma we compute $\mathfrak{D} = \det(\mathcal{R}_\Gamma)$ and show that it is $\neq 0$ for convex realizations.

Let us use the fact that most entries in $\mathcal{R}_\Gamma$ are zero. Observe that every 3×3 minor of $\mathcal{R}_\Gamma$ either contains a zero row or column, or two columns which add up to zero, or looks like

$$M(a \mid b, c, d) = \begin{pmatrix} x_a - x_b & y_a - y_b & z_a - z_b \\ x_a - x_c & y_a - y_c & z_a - z_c \\ x_a - x_d & y_a - y_d & z_a - z_d \end{pmatrix},$$

where a, b, c, d represent distinct integers in $[n]$. Here we assume that $b < c < d$ and the ordering on rows corresponding to edges $(v_i, v_j) \in E$ is lexicographic. In addition to these minors, there is one special non-degenerate 3×3 minor of $\mathcal{R}_\Gamma$, with rows corresponding to the edges (v_1, v_2), (v_1, v_3), and (v_2, v_3), and the columns to x_2, x_3, and y_3:

$$M(1, 2, 3) = \begin{pmatrix} x_2 & 0 & 0 \\ 0 & x_3 & y_3 \\ x_2 - x_3 & x_3 - x_2 & y_3 \end{pmatrix}.$$

Now, using the Laplace expansion for $\det \mathcal{R}_\Gamma$ over triples of rows we conclude that the determinant $\mathfrak{D}$ is the product of determinants of the 3×3 minors as above, each given up to a sign. Since we need these signs let us formalize this as follows.

We say that vertices v_1, v_2, v_3 are *base vertices* and the edges between them are *base edges*. A *claw* in Γ is a subgraph H of Γ isomorphic to $K_{1,3}$, i.e., a subgraph $K(a \mid b, c, d)$ consisting of four distinct vertices v_a, v_b, v_c, v_d and three edges: (v_a, v_b), (v_a, v_c), and (v_a, v_d). We call vertex v_a the *root* of the claw $K(a \mid b, c, d)$. Recall that Γ