§19. Excerpts from class, November 6 [notes by TLL]

We spent the first half of class examining the solutions to a homework assignment (see §20 below). Don says that the solutions were surprisingly good (see §21).

One of the proofs described in that section contains illustrations in four colors. Don says that color can be used effectively in talks, but usually not in papers (for that matter, Leslie Lamport says that proofs should never be presented in talks, but only in papers). Technical illustrations, even without four colors, cause no end of trouble: Don says that the amount of work involved in preparing a paper for publication is proportional to the cube of the number of illustrations. But they are indispensable in many cases.

Don showed us several of the illustrations, charts, and tables from *The Art of Computer Programming*, Volume 3, and recounted the difficulties in choosing clear methods of presenting his ideas. He also mentioned some technical and artistic problems that he had with an illustration: At what angle should the truncated octahedron on page 13 be displayed?

His books contain some numerical tables ("which are sometimes thought to be unenlightening"); Don says that they can sometimes present ideas that can't be demonstrated graphically (such as numbers oscillating about 2 with period 2π, page 41). Diagrams with accompanying text are also used. Don made sure that the final text was arranged opposite the diagrams to which it refers.

The book contains a running example of how 16 particular numbers are sorted by dozens of different algorithms. Each algorithm leads to a different graphical presentation of the sorting activities on those numbers (pages 77, 82, 84, 97, 98, 106, 110, 113, 115, 124, 140, 143, 147, 151, 161, 165, 166, 172, 175, 205, 251, 253, 254, 359).

§20. A Homework Problem

The Appendix to Gillman's book takes a paper that has horrible notation and simplifies it greatly. Your assignment is to take Gillman's simplification and produce something simpler yet. Aim for notation that needs no double subscripts or subscripted superscripts. This assignment will be graded! Please take time to do your best.

Here is a statement of Gillman's simplification. This is your starting point. What is the best way to present Sierpiński's theorem?

Lemma. *There is a one-to-one correspondence between the set of all real numbers α and the set of all pairs $(\langle n_k\rangle, \langle t_k\rangle)$, where $\langle n_k\rangle_{k\ge1}$ is an increasing sequence of positive integers and $\langle t_k\rangle_{k\ge1}$ is a sequence of real numbers.*

Notation. The sequences $\langle n_k\rangle$ and $\langle t_k\rangle$ corresponding to α are called $\langle n_k^\alpha\rangle$ and $\langle t_k^\alpha\rangle$. The set of real numbers is called $\mathbf{R}$.

Theorem. *Assume that $\langle A_\alpha \rangle_{\alpha \in \mathbf{R}}$ is a family of countably infinite subsets of $\mathbf{R}$ such that, for $\alpha \neq \beta$, either $\alpha \in A_\beta$ or $\beta \in A_\alpha$. Then there is a sequence of functions $f_n : \mathbf{R} \to \mathbf{R}$ such that, if S is any uncountable subset of $\mathbf{R}$, we have $f_n(S) = \mathbf{R}$ for all but finitely many f_n.*

Proof. Let the countable set A_α consist of the real numbers

$$\{\alpha_1, \alpha_2, \alpha_3, \ldots\}\,.$$

If α is any real number, define an increasing sequence of positive integers $\langle l_k^\alpha \rangle$ by letting $l_1^\alpha = n_1^{\alpha_1}$ and then, after l_{k-1}^α has been defined, letting l_k^α be the least integer in the sequence $\langle n_1^{\alpha_k}, n_2^{\alpha_k}, \ldots \rangle$ that is greater than l_{k-1}^α.

Let f_n be the function on real numbers defined by the rule

$$f_n(\alpha) = \begin{cases} t_n^{\alpha_k}\,, & \text{if } n = l_k^\alpha \text{ for some } k \geq 1\,; \\ \alpha\,, & \text{otherwise.} \end{cases}$$

We will show that the sequence of functions f_n satisfies the theorem, by proving that any set S for which infinitely many n have $f_n(S) \neq \mathbf{R}$ must be countable.

Suppose, therefore, that $\langle n_k \rangle$ is an increasing sequence of integers and that $\langle t_k \rangle$ is a sequence of real numbers such that

$$t_{n_k} \notin f_{n_k}(S)\,, \qquad \text{for all } k \geq 1\,.$$

Let $t_j = 0$ if j is not one of the numbers $\{n_1, n_2, \ldots\}$. By the lemma, there's a real number β such that $n_k = n_k^\beta$ and $t_k = t_k^\beta$ for all k.

Let α be any real number $\neq \beta$ such that $\alpha \notin A_\beta$. We will prove that $\alpha \notin S$; this will prove the theorem, because all elements of S must then lie in the countable set $A_\beta \cup \{\beta\}$.

By hypothesis, $\beta \in A_\alpha$. Hence we have $\beta = \alpha_k$ for some k. If we set $n = l_k^\alpha$, we know by the definition of f_n that

$$f_n(\alpha) = t_n^{\alpha_k} = t_n^\beta = t_n\,.$$

But the construction of l_k^α tells us that $n = n_j^{\alpha_k} = n_j^\beta = n_j$ for some j. Therefore

$$f_{n_j}(\alpha) = t_{n_j}\,.$$

We chose $t_{n_j} \notin f_{n_j}(S)$, hence $\alpha \notin S$. ∎

[Here are additional excerpts from TLL's classnotes for October 16, when the homework problem was handed out:] The first thing that we learned in class today was that now would be a good time to buy Leonard Gillman's book (*Writing Mathematics Well*). Not only have several copies (finally) arrived at the bookstore, but Don has given us a homework assignment straight out of the Appendix of this book.

The assignment (which is due on Friday, October 30th) is to take the "simplified version" of the proof in Gillman's case study and to simplify it still further. The main simplifying principle is to minimize subscripts and superscripts. When we are done, there should be no subscripted subscripts and no subscripted superscripts. As Don said, "Try to recast the proof so that the idea of the proof remains the same, but the proof gets shorter."

The original proof was written by Sierpiński. Don told us that Sierpiński was a great mathematician who wrote several papers cited in *Concrete Mathematics*, from the year 1909 as well as 1959. But the notation in Sierpiński's original proof quoted by Gillman was so complicated that it confused even him: His proof contained an error that was found by another mathematician (after publication).

While the mathematics used in the proof is not trivial, it uses only functions and sets and should be accessible to us. (This is not to say that it is immediately obvious.) Anyone who is uncomfortable with what sets are, what it means for a set to be countable, or what a one-to-one correspondence is, may need some help with this assignment. Don recommended visiting the TAs during office hours as a good first step for those who feel they need help. (It might also help to remember that Don says, "It's not necessary to understand the proof completely in order to do this assignment.")

Don't worry if the hypothesis of the theorem seems pretty wild; it is pretty wild. It implies the "Continuum Hypothesis." The Continuum Hypothesis states that there are no infinities between the countably infinite (the cardinality of the integers) and the continuum (the cardinality of the real numbers). From 1900 to 1960, the truth or falsity of the Continuum Hypothesis was one of the most famous unsolved problems of mathematics; Sierpiński published his paper as a step toward solving that problem. Kurt Gödel proved in 1938 that the Continuum Hypothesis is consistent with standard set theory; Paul Cohen of Stanford proved 25 years later that the negation of the Continuum Hypothesis is also consistent. Thus we know now that the hypothesis can be neither proved nor disproved.

[Here are additional excerpts from PMR's classnotes for October 23:] The homework assignment is due a week from today, Don said; so do it as well as possible, and let's not have any excuses!