Abstract For a given set of line segments and a polygon P in the plane, we study the problem to find the maximum number of segments that can be disjointly embedded by translation into P. We show APX-hardness of this problem and discuss variations.

This problem can be considered in two respects: as a variant of the Kakeya problem and as a maximum-packing problem for line segments.

1 Introduction

The Kakeya Problem. The famous Kakeya problem asks for the region R in the plane with minimum area such that a unit-length line segment can continuously rotate by π within R. One variant of the Kakeya problem relaxes the continuous rotation and try to find a planar region R' with the minimum area such that translates of all the unit-length line segments in the plane can be placed in R'. The segments may intersect. This region R' is called a minimum area translation cover.

Pál [5, 4] solved these two problems, and many other interesting variations about the minimum-area translation cover have been studied (refer [6, 3] for surveys).

A Minimum-Container Problem and a 3-approximation Algorithm. Finding a minimum-area translation cover can be considered as a minimum-container problem if we want to *disjointly* embed line segments. The following question arises naturally in this context; given a set of line segments $\mathcal{S}$, what is the minimum-area convex body R such that translates of segments in $\mathcal{S}$ can be disjointly embedded in R?

A 3-approximation algorithm for this problem is as follows. ...

Problem Definition and Summary of Results. To solve a minimum-container problem it is natural to consider its dual, that is, a maximum-packing problem. We consider the maximum-packing problem in this abstract. We show hardness results for a simple polygon and a simple approximation algorithm for a convex polygon.

As in [2], we define MaxSegPack$d$ for a class $\mathcal{R}$ of regions in $\mathbb{R}^d$ as the following problem; given a collection of (open) segments and a region $R \in \mathcal{R}$, what is the maximum number of segments that can be disjointly embedded in R by translation?

Before describing the reduction from MAX-3-SAT, we state the following two lemmas for constructing gadgets.

Lemma 1. *Let S be a set of unit-length line segments with distinct slopes. There exists a convex polygon $Q = Q(S)$ with the following properties:*

1. *any segment $s \in S$ fits in Q;*
2. *no two segments in S can be packed in Q; and*
3. *no unit-length line segment $s \notin S$ fits in Q.*

Proof. Translate all the segments of S so that their midpoints lie at the origin. Now define $Q(S)$ as the convex hull of all those segments.

The diameter of Q is 1 and the diameter is attained only for pairs of points that lie at the opposite extreme points of Q. Therefore, a unit-length line segment s fits in Q if and only if s can be translated in a way that its endpoints lie at the opposite extreme points of Q. This implies the first and the third property.

Each segment s that fits in Q has a unique position in Q and this unique position always goes through the origin. Thus, no two segments of unit-length can be packed in Q. This implies the second property. □

Lemma 2. *Let S be a set of unit-length line segments such that the angle with the x-axis is within ± 0.1 radiant. And let S' be a set of unit-length line segments such that the angle with the y-axis is within ± 0.1 radiant.*

There exists a convex polygon $R = R(S, S')$ with the following properties:

1. *segments in S can be packed in R;*
2. *segments in S' can be packed in R;*
3. *no segments $s \in S$ and $s' \in S'$ can be embedded simultaneously in R; and*
4. *no unit segment $s \notin S \cup S'$ fits into R.*

Proof. Translate the left endpoint of every line segment $s \in S$ to the point $(-0.5, 0)$ and the bottom endpoint of every line segment $s' \in S'$ to the point $(0, -0.5)$. The convex hull of those segments define $R = R(S, S')$.

The diameter of Q is 1 and the diameter is attained only for pairs of points (p, q) such that either 1) p is at $(-0.5, 0)$ and q is one of right extreme points. or 2) p is at $(0, -0.5)$ and q is one of top extreme points. These are exactly the endpoints of segments in $S \cup S'$ after we moved the segments of S. By the same argument as in Lemma 1, any unit-length line segment s fits in R if and only if $s \in S \cup S'$. Each segment s that fits in R has a unique position $p(s)$ in R. Observe that $p(s_1)$ and $p(s_2)$ are disjoint if either $s_1, s_2 \in S$ or $s_1, s_2 \in S'$ and $p(s_1)$ and $p(s_2)$ intersect otherwise. Thus, any two segments s_1 and s_2 can be embedded simultaneously in R if and only if either $s_1, s_2 \in S$ or $s_1, s_2 \in S'$. Altogether these arguments imply the above four properties. □