This is one case of the "Master Theorem" for divide-and-conquer recurrences.

Theorem 1. *Let $a \geq 1$ and $b > 1$ be constants, and let $f(n)$ and $T(n)$ be nonnegative functions defined on the nonnegative integers. Let $\gamma := \log_b a > 0$, and let $n_0 \geq 1$ be some integer constant.*

(a) *Suppose we have the recurrence inequality*

$$T(n) \leq aT(\lceil n/b \rceil) + f(n), \text{ for all } n > n_0. \tag{1}$$

If $f(n) = O(n^\gamma)$, then

$$T(n) = O(n^\gamma \log n).$$

(b) *Suppose we have the recurrence inequality*

$$T(n) \geq aT(\lfloor n/b \rfloor) + f(n), \text{ for all } n > n_0. \tag{2}$$

If $f(n) = \Omega(n^\gamma)$, then

$$T(n) = \Omega(n^\gamma \log n).$$

(c) *If the recurrence* (2) *holds and $T(n) > 0$ for all n, then*

$$T(n) = \Omega(n^\gamma).$$

The proof will be given in the style proposed by E. W. Dijkstra[1], but without motivation or explanation.

Proof. (a) The upper bound. Set

$$V := b/(b-1). \tag{3}$$

By arithmetic manipulations, we can conclude that

$$-V/b = 1 - V. \tag{4}$$

If necessary, increase the integer threshold n_0 such that

$$n_0/b \geq V + 2 \tag{5}$$

and

$$n_0/b \leq n_0 - 1. \tag{6}$$

[1] Edsger W. Dijkstra. *The notational conventions I adopted, and why.* EWD1300, July 2000. `https://www.cs.utexas.edu/users/EWD/transcriptions/EWD13xx/EWD1300.html`

Increasing n_0 makes the assumption (1) weaker, and thus we can assume (5–6) without loss of generality. From the last inequality, we obtain

$$n/b \leq n - 1, \text{ for all } n \geq n_0. \tag{7}$$

Set

$$M := \max\{T(0), T(1), T(2), \ldots, T(n_0)\} \tag{8}$$

By the assumption $f(n) = O(n^\gamma)$, we can choose u such that

$$f(n) \leq u \cdot n^\gamma, \tag{9}$$

for all $n \geq 1$.

$$u' := \max\left\{u\left(\frac{n_0}{n_0 - V}\right)^\gamma, \frac{M}{(n_0/b - V)^\gamma \log_b(n_0/b - V)}\right\}, \tag{10}$$

which is well-defined and positive due to (5). It follows that

$$u \cdot n^\gamma \leq u' \cdot (n - V)^\gamma, \text{ for all } n \geq n_0 \tag{11}$$

We define the nonnegative function

$$\hat{T}(n) := u'(n - V)^\gamma \log_b(n - V) \tag{12}$$

for all real numbers $n \geq V + 1$. It is a product of two nonnegative increasing functions and is therefore monotone increasing. We claim that $\hat{T}(n)$ is an upper bound on $T(n)$:

$$T(n) \leq \hat{T}(n), \text{ for all integers } n \geq n_0/b \tag{13}$$

From this, the desired asymptotic bound $T(n) = O(n^\gamma \log n)$ follows immediately. Note that, by (5), the interval $[n_0/b, \infty)$ for n is contained in the domain $[V + 1, \infty)$ of $\hat{T}$, and therefore the inequality (13) makes sense.

As induction basis, we prove (13) for the range $n_0/b \leq n \leq n_0$ directly:

$\quad \hat{T}(n)$

$\geq \quad \{ \hat{T}$ is monotone increasing $\}$

$\quad \hat{T}(n_0/b)$

$= \quad \{$ definition of $\hat{T}$ $\}$

$\quad u'(n_0/b - V)^\gamma \log_b(n_0/b - V)$

$\geq \quad \{$ second term in the definition (10) of u' $\}$

$\quad M$

$\geq \quad \{$ definition (8) of M, assumption $n \leq n_0$ $\}$

$\quad T(n)$

We are ready for the induction step to prove (13). We assume $n > n_0$, and the induction hypothesis is that $T(i) \leq \hat{T}(i)$ has been proved for all i in the range $n_0/b \leq i < n$.

$$
\begin{array}{ll}
 & T(n) \\
\leq & \quad \{ \text{ recurrence (1) } \} \\
 & aT(\lceil n/b \rceil) + f(n) \\
\leq & \quad \{ \lceil n/b \rceil < n \text{ by (7), } \lceil n/b \rceil \geq n_0/b \text{, induction hypothesis } \} \\
 & a\hat{T}(\lceil n/b \rceil) + f(n) \\
\leq & \quad \{ \hat{T} \text{ is monotone } \} \\
 & a\hat{T}(n/b + 1) + f(n) \\
\leq & \quad \{ \text{ definition (12) of } \hat{T} \} \\
 & au'(n/b + 1 - V)^\gamma \log_b(n/b + 1 - V) + f(n) \\
\leq & \quad \{ \text{ (4) } \} \\
 & au'(n/b - V/b)^\gamma \log_b(n/b - V/b) + f(n) \\
= & \quad \{ \text{ rearrangement } \} \\
 & (a/b^\gamma)u'(n - V)^\gamma \log_b((n - V)/b) + f(n) \\
= & \quad \{ a = b^\gamma \text{ by the definition of } \gamma \text{, laws of logarithms } \} \\
 & u'(n - V)^\gamma \big(\log_b(n - V) - 1\big) + f(n) \\
\leq & \quad \{ f(n) = O(n^\gamma) \text{, condition (9) on } u \} \\
 & u'(n - V)^\gamma \big(\log_b(n - V) - 1\big) + u \cdot n^\gamma \\
\leq & \quad \{ \text{ (11) } \} \\
 & u'(n - V)^\gamma \big(\log_b(n - V) - 1\big) + u' \cdot (n - V)^\gamma \\
= & \quad \{ \text{ arithmetic } \} \\
 & u'(n - V)^\gamma \log_b(n - V) \\
= & \quad \{ \text{ definition of } \hat{T} \} \\
 & \hat{T}(n)
\end{array}
$$

(b) The lower bound for the assumption $f(n) = \Omega(n^\gamma)$. This is in many ways analogous to part (a). By the assumption $f(n) = \Omega(n^\gamma)$, there are constants $u > 0$ and n_1 such that

$$f(n) \geq u \cdot n^\gamma, \text{ for all } n \geq n_1 \tag{14}$$

It follows directly from (2) that $T(n) \geq f(n)$, and hence

$$T(n) > 0, \text{ for } n \geq n_1. \tag{15}$$

We use the same definition (3) of V as in part (a), and we impose the same constraints (5–6) on n_0. In addition, we also require that n_0 is big enough to fulfill the inequality

$$n_0/b - 1 \geq n_1. \tag{16}$$

Set

$$m := \min\{T(\lfloor n_0/b \rfloor), \dots, T(n_0 - 1), T(n_0)\} \tag{17}$$

By (15) and (16),

$$m > 0. \tag{18}$$

Set

$$u' := \min\left\{ u \left(\frac{n_0}{n_0 + V} \right)^\gamma, \frac{m}{(n_0 + V)^\gamma \log_b(n_0 + V)} \right\}, \tag{19}$$

which is positive due to (18). We define the nonnegative function

$$\hat{T}(n) := u'(n + V)^\gamma \log_b(n + V) \tag{20}$$

for all real numbers $n \geq 0$. Since $V > 1$, by (3), this is a product of two nonnegative increasing functions and is therefore monotone increasing. We claim that $\hat{T}(n)$ is a lower bound on $T(n)$:

$$T(n) \geq \hat{T}(n), \text{ for all integers } n > n_0/b - 1 \tag{21}$$

From this, the desired asymptotic bound $T(n) = \Omega(n^\gamma \log n)$ follows immediately. Note that, by 5, $n_0/b > 1$, and the range of n in which we claim (21) is contained in the domain of $\hat{T}$. As induction basis, we prove (21) for the range $n_0/b - 1 < n \leq n_0$ directly:

$\quad \hat{T}(n)$
$\leq \qquad \{ \hat{T}$ is monotone increasing $\}$
$\quad \hat{T}(n_0)$
$= \qquad \{$ definition of $\hat{T}$ $\}$
$\quad u'(n_0 + V)^\gamma \log_b(n_0 + V)$
$\leq \qquad \{$ second term in the definition (19) of u' $\}$
$\quad m$
$\leq \qquad \{$ definition (17) of m, assumption $n_0/b - 1 < n \leq n_0$ $\}$
$\quad T(n)$

We are ready for the induction step to prove (21). Assume $n > n_0$. The induction hypothesis is that $T(i) \geq \hat{T}(i)$ has been proved for all i in the range $n_0/b - 1 < i < n$.

$T(n)$

$\geq$ $\{$ recurrence (2) $\}$

$aT(\lfloor n/b \rfloor) + f(n)$

$\geq$ $\{$ $n_0/b - 1 < \lfloor n/b \rfloor$ and $\lfloor n/b \rfloor < n$ by (7), induction hypothesis $\}$

$a\hat{T}(\lfloor n/b \rfloor) + f(n)$

$\geq$ $\{$ $\hat{T}$ is monotone $\}$

$a\hat{T}(n/b - 1) + f(n)$

$\geq$ $\{$ definition (20) of $\hat{T}$ $\}$

$au'(n/b - 1 + V)^\gamma \log_b(n/b - 1 + V) + f(n)$

$\geq$ $\{$ (4) $\}$

$au'(n/b + V/b)^\gamma \log_b(n/b + V/b) + f(n)$

$=$ $\{$ rearrangement $\}$

$(a/b^\gamma)u'(n + V)^\gamma \log_b((n + V)/b) + f(n)$

$=$ $\{$ $a = b^\gamma$ by the definition of γ, laws of logarithms $\}$

$u'(n + V)^\gamma \big(\log_b(n + V) - 1\big) + f(n)$

$\geq$ $\{$ condition (14) on u, $n > n_0 \geq n_1$ by (16) $\}$

$u'(n + V)^\gamma \big(\log_b(n + V) - 1\big) + u \cdot n^\gamma$

$\geq$ $\{$ first term in the definition (19) of u', $n \geq n_0$ $\}$

$u'(n + V)^\gamma \big(\log_b(n + V) - 1\big) + u' \cdot (n + V)^\gamma$

$=$ $\{$ arithmetic $\}$

$u'(n + V)^\gamma \log_b(n + V)$

$=$ $\{$ definition of $\hat{T}$ $\}$

$\hat{T}(n)$ □

(c) We finally prove the lower bound for the recursion (2) without any assumption on f. We only require that T is positive. Set

$$m := \min\{T(0), T(1), T(2), \ldots, T(n_0 - 1), T(n_0)\}. \tag{22}$$

By assumption,

$$m > 0, \tag{23}$$

and hence the constant

$$u' := \frac{m}{(n_0 + V)^\gamma} \tag{24}$$

is also positive, where V is still the same constant defined above (3). We define the nonnegative and monotone increasing function

$$\hat{T}(n) := u'(n + V)^\gamma \tag{25}$$

for all real numbers $n \geq 0$. We claim that $\hat{T}(n)$ is a lower bound on $T(n)$:

$$T(n) \geq \hat{T}(n), \text{ for all } n \geq 0 \tag{26}$$

From this, the desired asymptotic bound $T(n) = \Omega(n^\gamma)$ follows immediately.

As induction basis, we prove (26) for $0 \leq n \leq n_0$ directly:

$$
\begin{array}{ll}
 & \hat{T}(n) \\
\leq & \quad \{ \hat{T} \text{ is monotone increasing } \} \\
 & \hat{T}(n_0) \\
= & \quad \{ \text{ definition of } \hat{T} \} \\
 & u'(n_0 + V)^\gamma \\
\leq & \quad \{ \text{ definition (24) of } u' \} \\
 & m \\
\leq & \quad \{ \text{ definition (22) of } m, \text{ assumption } n \leq n_0 \} \\
 & T(n)
\end{array}
$$

For the induction step, we consider $n > n_0$, and the induction hypothesis is that $T(i) \geq \hat{T}(i)$ has been proved for all $i < n$.

$$
\begin{array}{ll}
 & T(n) \\
\geq & \quad \{ \text{ recurrence (2) } \} \\
 & aT(\lfloor n/b \rfloor) \\
\geq & \quad \{ \lfloor n/b \rfloor < n, \text{ induction hypothesis } \} \\
 & a\hat{T}(\lfloor n/b \rfloor) \\
\geq & \quad \{ \hat{T} \text{ is monotone } \} \\
 & a\hat{T}(n/b - 1) \\
\geq & \quad \{ \text{ definition (25) of } \hat{T} \} \\
 & au'(n/b - 1 + V)^\gamma \\
\geq & \quad \{ \text{ (4) } \} \\
 & au'(n/b + V/b)^\gamma \\
= & \quad \{ \text{ rearrangement } \} \\
 & (a/b^\gamma)u'(n + V)^\gamma \\
= & \quad \{ a = b^\gamma \text{ by the definition of } \gamma \} \\
 & u'(n + V)^\gamma \\
= & \quad \{ \text{ definition of } \hat{T} \} \\
 & \hat{T}(n)
\end{array}
$$

□